THE THEORY OF COUPLED CIRCUITS.

By Louis Cohen.

INTRODUCTION.

Professor Braun,[1] in seeking for some method whereby he could increase the amount of energy available for radiation in wireless telegraphy, without employing excessively high voltages, conceived the idea of using coupled circuits. The advantages of this method over the old method—direct excited antennæ—are quite obvious, and are familiar to all students of this subject. By the use of an auxiliary circuit of low resistance and large capacity, we can store up a large amount of energy, and at the same time produce a more prolonged oscillation, which is very important for syntonization.

The coupling may be accomplished in two different ways, electromagnetically or direct, as indicated in Figs. 1 and 2. In either case the advantages to be gained are the same.

Owing to its great practical importance, several able investigators have worked on this problem, and we find in the literature of the subject during the past few years a number of contributions dealing with this problem from its practical as well as theoretical aspect.
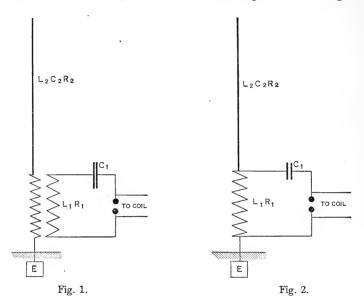
This problem finds its analogy in accoustics. If we mount several tuning forks on a resonating box and excite one, the other tuning forks will also vibrate, but each one will vibrate with several distinct frequencies which are different from the natural period of the tuning forks. The general outline of the problem and the method of its solution were given by Lord Rayleigh.[2]

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[1] F. Braun, Physikalische Zs. **3**, p. 148; 1901.
[2] Lord Rayleigh: Theory of Sound, Vol. **1**, p. 84.

He points out that the whole motion may be resolved into m normal harmonic vibrations of different periods, each of which is entirely independent of the others.

If we denote the displacement of the respective tuning forks by

Fig. 1.　　　　　　　　　　　　　Fig. 2.

ψ_1, ψ_2, ... ψ_m, and neglect the frictional resistance, then we may write:

$$\psi_1 = A_1 \cos (n_1 t - a) + A_2 \cos (n_2 t - \beta) + A_3 \cos (n_3 t - \gamma) + \ . \ . \ . \ . \ .$$

$$\psi_2 = B_1 \cos (n_1 t - a) + B_2 \cos (n_2 t - \beta) + B_3 \cos (n_3 t - \gamma) + \ . \ . \ .$$

$$. \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ .$$

$$\psi_m = C_1 \cos (n_1 t - a) + C_2 \cos (n_2 t - \beta) + C_3 \cos (n_3 t - \gamma) + \ . \ . \ .$$

where n_1^2, n_2^2, etc., are the m roots of an equation of mth degree in n^2.

If we neglect the resistance, the electrical problem under consideration becomes a very simple matter, for being a system of only two degrees of freedom we shall have only two frequencies, and this will require obtaining the roots of a second-degree equation. If, however, we take the resistance into consideration, which is a very important factor in this case, then

the problem becomes much more difficult, for under these conditions we shall have to obtain the roots of a biquadratic instead of a quadratic, as can be seen in the following way. Writing down the differential equations for the potentials in the two circuits, we have:

$$L_1 C_1 \frac{d^2 V_1}{dt^2} + R_1 C_1 \frac{dV_1}{dt} + V_1 + M C_2 \frac{d^2 V_2}{dt^2} = 0$$

$$L_2 C_2 \frac{d^2 V_2}{dt^2} + R_2 C_2 \frac{dV_2}{dt} + V_2 + M C_1 \frac{d^2 V_1}{dt^2} = 0$$

The derivations of these two equations will be given later.

Now assume:

$$V_1 = A e^{\lambda t}$$

$$V_2 = B e^{\lambda t}$$

Introducing the values of V_1 and V_2 in the above equations and eliminating $\frac{A}{B}$, we obtain the following equation:

$$C_1 C_2 (L_1 L_2 - M^2) \lambda^4 + C_1 C_2 (R_1 L_2 + R_2 L_1) \lambda^3 + (L_1 C_1 + L_2 C_2 + R_1 R_2 C_1 C_2) \lambda^2 + (R_1 C_1 + R_2 C_2) \lambda + 1 = 0$$

and the solution of our problem reduces itself to the determination of the roots of the biquadratic.

Theoretically, of course, it is possible to obtain the roots of a fourth-degree equation; but practically the task is not an easy one, and this is more so in this problem when the coefficients happen to be somewhat complicated expressions.

The problem of electromagnetically coupled circuits is from the mathematical standpoint precisely the same as that of the Tesla transformer, and a solution of the problem was attempted some thirteen years ago by Oberbeck,[3] Galitzin,[4] and Domalys and Kolacek.[5] Later on, very able contributions to the subject were made by Wien[6] and Drude.[7] All these investigations have in general fol-

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[3] A. Oberbeck, Wied. Ann. **55**, p. 623; 1895.
[4] Fürst B. Galitzin, Petersb. Ber., May, June; 1895.
[5] R. Domalys and F. Kolacek, Wied. Ann. **57**, p. 731; 1896.
[6] M. Wien, Wied. Ann. **61**, p. 151; 1897.
[7] P. Drude, Ann. der Physik. **13**, p. 512; 1904.

lowed the method indicated by Lord Rayleigh, and to avoid the difficulty of obtaining the roots of a biquadratic, they were obliged to limit themselves to consider some special cases only, or at the best get some approximate results. It may also be noted that the method that has been adopted by these investigators offer considerable difficulties in the evaluation of the constants of integration. Furthermore, all discussions of the problem were limited to the case of electromagnetic coupling; and to my knowledge the problem of direct coupled circuits has not been considered at all, except the very special case of no resistance which has been developed by Seibt,[8] and of course if we neglect the resistance, we eliminate from our discussion the most important part, namely, the determination of the damping factors.

In this paper I have developed a method which enables me to obtain the complete solution to the problem, and yet avoid the difficulty of getting the roots of a biquadratic equation. My method is applicable to the problem of direct coupled circuits as well as the electromagnetic coupled circuits, and the evaluation of the constants can be obtained with considerable ease. I have worked out both cases quite fully, obtaining the frequency constants as well as the damping factors. A discussion of the results and the comparative merits of the two systems will be given later, and I trust that the results obtained will be of interest to those engaged in the work of wireless telegraphy and telephony.

ELECTROMAGNETICALLY COUPLED CIRCUITS.

Let us consider two circuits, each having resistance, R, inductance, L, and capacity, C, and specify these quantities for the two circuits by suffixes 1 and 2. Let these two circuits have a mutual inductance, M, then denote by I_1 and I_2 the respective currents at any instant, v_1 and v_2 the respective potentials of the condensers of the two circuits, and we have the following fundamental equations:

$$\left. \begin{aligned} L_1 \frac{dI_1}{dt} + R_1 I_1 + v_1 + M \frac{dI_2}{dt} &= 0 \\[2mm] L_2 \frac{dI_2}{dt} + R_2 I_2 + v_2 + M \frac{dI_1}{dt} &= 0 \end{aligned} \right\} \tag{1}$$

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[8] G. Seibt, Physik. Zs., Aug. 1, 1904.

We have also the relations:

$$I_1 = C_1 \frac{dv_1}{dt}, \qquad I_2 = C_2 \frac{dv_2}{dt}.$$

Introducing these values in (1) we obtain:

$$L_1 C_1 \frac{d^2 v_1}{dt^2} + R_1 C_1 \frac{dv_1}{dt} + v_1 + M C_2 \frac{d^2 v_2}{dt^2} = 0$$

$$L_2 C_2 \frac{d^2 v_2}{dt^2} + R_2 C_2 \frac{dv_2}{dt} + v_2 + M C_1 \frac{d^2 v_1}{dt^2} = 0$$

or using the following abbreviations:

$$\left. \begin{aligned} a_1 &= L_1 C_1, & b_1 &= R_1 C_1, & d_1 &= M C_2 \\ a_2 &= L_2 C_2, & b_2 &= R_2 C_2, & d_2 &= M C_1 \end{aligned} \right\} \tag{2}$$

We may write the above equations in the following form:

$$\left. \begin{aligned} a_1 \frac{d^2 v_1}{dt^2} + b_1 \frac{dv_1}{dt} + v_1 + d_1 \frac{d^2 v_2}{dt^2} = 0 \\ a_2 \frac{d^2 v_2}{dt^2} + b_2 \frac{dv_2}{dt} + v_2 + d_2 \frac{d^2 v_1}{dt^2} = 0 \end{aligned} \right\} \tag{3}$$

Let us assume:

$$\left. \begin{aligned} v_1 &= m_1 w_1 + m_2 w_2 \\ v_2 &= m_3 w_1 + m_4 w_2 \end{aligned} \right\} \tag{4}$$

where m_1, m_2, m_3, m_4 are constants, and w_1, w_2 are different functions of t. Introducing the values of v_1 and v_2 as given by (4) into (3) we obtain:

$$\left. \begin{aligned} (a_1 m_1 + d_1 m_3) \frac{d^2 w_1}{dt^2} + (a_1 m_2 + d_1 m_4) \frac{d^2 w_2}{dt^2} + b_1 m_1 \frac{dw_1}{dt} + b_1 m_2 \frac{dw_2}{dt} + \\ m_1 w_1 + m_2 w_2 = 0 \\[2mm] (a_2 m_3 + d_2 m_1) \frac{d^2 w_1}{dt^2} + (a_2 m_4 + d_2 m_2) \frac{d^2 w_2}{dt^2} + b_2 m_3 \frac{dw_1}{dt} + b_2 m_4 \frac{dw_2}{dt} + \\ m_3 w_1 + m_4 w_2 = 0 \end{aligned} \right\} \tag{5}$$

Eliminating from these two equations, first the term containing

$$\frac{d^2w_1}{dt^2} \text{ and then the term containing } \frac{d^2w_2}{dt^2},$$

we shall obtain the following two equations:

$$\left.
\begin{aligned}
&\left\{(a_1m_2+d_1m_4)(a_2m_3+d_2m_1)-(a_2m_4+d_2m_2)(a_1m_1+d_1m_3)\right\}\frac{d^2w_2}{dt^2}+\\
&\left\{b_1m_1(a_2m_3+d_2m_1)-b_2m_3(a_1m_1+d_1m_3)\right\}\frac{dw_1}{dt}+\\
&\left\{b_1m_2(a_2m_3+d_2m_1)-b_2m_4(a_1m_1+d_1m_3)\right\}\frac{dw_2}{dt}+\\
&\left\{m_1(a_2m_3+d_2m_1)-m_3(a_1m_1+d_1m_3)\right\}w_1+\\
&\left\{m_2(a_2m_3+d_2m_1)-m_4(a_1m_1+d_1m_3)\right\}w_2=0
\end{aligned}
\right\} \quad (6)$$

$$\left.
\begin{aligned}
&\left\{(a_2m_4+d_2m_2)(a_1m_1+d_1m_3)-(a_1m_2+d_1m_4)(a_2m_3+d_2m_1)\right\}\frac{d^2w_1}{dt^2}+\\
&\left\{b_1m_1(a_2m_4+d_2m_2)-b_2m_3(a_1m_2+d_1m_4)\right\}\frac{dw_1}{dt}+\\
&\left\{b_1m_2(a_2m_4+d_2m_2)-b_2m_4(a_1m_2+d_1m_4)\right\}\frac{dw_2}{dt}+\\
&\left\{m_1(a_2m_4+d_2m_2)-m_3(a_1m_2+d_1m_4)\right\}w_1+\\
&\left\{m_2(a_2m_4+d_2m_2)-m_4(a_1m_2+d_1m_4)\right\}w_2=0
\end{aligned}
\right\} \quad (7)$$

Let us now give to the constants m_1, m_2, m_3, m_4 such values as to make the coefficients of w_1 in equation (6) and the coefficient of w_2 in equation (7) separately equal to zero; that is, put:

$$a_2m_1m_3+d_2m_1{}^2-a_1m_1m_3-d_1m_3{}^2=0$$

$$a_2m_2m_4+d_2m_2{}^2-a_1m_2m_4-d_1m_4{}^2=0$$

From which we get:

$$\frac{m_1}{m_3}=\frac{m_2}{m_4}=\frac{a_1-a_2\pm\sqrt{(a_1-a_2)^2+4d_1d_2}}{2d_2}.$$

Using different signs before the radical for the different ratios, we get:

$$\frac{m_1}{m_3} = k_1 = \frac{a_1 - a_2 + \sqrt{(a_1 - a_2)^2 + 4d_1 d_2}}{2d_2}$$

$$\frac{m_2}{m_4} = k_2 = \frac{a_1 - a_2 - \sqrt{(a_1 - a_2)^2 + 4d_1 d_2}}{2d_2}$$

(8)

Expanding now the various coefficients in equations (6) and (7) and dividing through by $m_3 m_4$, we get:

$$(a_1 a_2 - d_1 d_2)(k_2 - k_1)\frac{d^2 w_2}{dt^2} + (a_2 b_1 k_2 + b_1 d_2 k_1 k_2 - a_1 b_2 k_1 - b_2 d_1)\frac{dw_2}{dt} +$$

$$(a_2 k_2 + d_2 k_1 k_2 - a_1 k_1 - d_1) w_2 +$$

$$\frac{m_1}{m_4}\left(a_2 b_1 + d_2 b_1 k_1 - a_1 b_2 - \frac{b_2 d_1}{k_1}\right)\frac{dw_1}{dt} = 0$$

$$(a_1 a_2 - d_1 d_2)(k_1 - k_2)\frac{d^2 w_1}{dt^2} + (a_2 b_1 k_1 + b_1 d_2 k_1 k_2 - a_1 b_2 k_2 - b_2 d_1)\frac{dw_1}{dt} +$$

$$(a_2 k_1 + d_2 k_1 k_2 - a_1 k_2 - d_1) w_1 +$$

$$\frac{m_2}{m_3}\left(a_2 b_1 + b_1 d_2 k_2 - a_1 b_2 - \frac{d_1 b_2}{k_2}\right)\frac{dw_2}{dt} = 0.$$

(9)

Suppose we put now for brevity $\dfrac{m_4}{m_1} = k_3$ and

$$\frac{m_2}{m_3} = \frac{m_4}{m_1}\frac{m_1}{m_4}\frac{m_2}{m_3} = k_3 k_1 k_2.$$

Then we may write equations (9) in the following form:

$$P_1 \frac{d^2 w_2}{dt^2} + P_2 \frac{dw_2}{dt} + P_3 w_2 + \frac{1}{k_3} P_4 \frac{dw_1}{dt} = 0$$

$$T_1 \frac{d^2 w_1}{dt^2} + T_2 \frac{dw_1}{dt} + T_3 w_1 + k_3 T_4 \frac{dw_2}{dt} = 0$$

(10)

where:

$$P_1 = (a_1 a_2 - d_1 d_2)(k_2 - k_1) \qquad T_1 = (a_1 a_2 - d_1 d_2)(k_1 - k_2) = -P_1$$

$$P_2 = a_2 b_1 k_2 + b_1 d_2 k_1 k_2 - a_1 b_2 k_1 \qquad T_2 = a_2 b_1 k_1 + b_1 d_2 k_1 k_2 - a_1 b_2 k_2$$
$$\qquad - b_2 d_1 \qquad\qquad\qquad\qquad\qquad - b_2 d_1$$

$$P_3 = a_2 k_2 + d_2 k_1 k_2 - a_1 k_1 - d_1 \qquad T_3 = a_2 k_1 + d_2 k_1 k_2 - a_1 k_2 - d_1$$

$$P_4 = a_2 b_1 + d_2 b_1 k_1 - a_1 b_2 - \frac{b_2 d_1}{k_1} \qquad T_4 = k_1 k_2 \left(a_2 b_1 + b_1 d_2 k_2 - a_1 b_2 - \frac{b_2 d_1}{k_2}\right)$$

(11)

Let us assume now that,

$$w_1 = A e^{-at} w_1' \left.\vphantom{\begin{matrix}a\\b\end{matrix}}\right\}$$
$$w_2 = B e^{-at} w_2' \tag{12}$$

where w_1' and w_2' are functions of t, and a is an arbitrary constant, we have,

$$\frac{dw_1}{dt} = A e^{-at} \frac{dw_1'}{dt} - aA e^{-at} w_1'$$

$$\frac{d^2 w_1}{dt^2} = A e^{-at} \frac{d^2 w_1'}{dt^2} - 2A a e^{-at} \frac{dw_1'}{dt} + A a^2 e^{-at} w_1'$$

and similar expressions for $\dfrac{dw_2}{dt}$ and $\dfrac{d^2 w_2}{dt^2}$.

Introducing these values in equations (10) we get:

$$\left.\begin{aligned}
&\left\{ P_1 \frac{d^2 w_2'}{dt^2} - 2P_1 a \frac{dw_2'}{dt} + P_1 a^2 w_2' + P_2 \frac{dw_2'}{dt} - P_2 a w_2' + \right. \\
&\left. P_3 w_2' \right\} B e^{-at} + \frac{1}{k_3} \left(P_4 \frac{dw_1'}{dt} - P_4 a w_1' \right) A e^{-at} = 0 \\
&\left\{ T_1 \frac{d^2 w_1'}{dt^2} - 2T_1 a \frac{dw_1'}{dt} + T_1 a^2 w_1' + T_2 \frac{dw_1'}{dt} - T_2 a w_1' + \right. \\
&\left. T_3 w_1' \right\} A e^{-at} + k_3 \left(T_4 \frac{dw_2'}{dt} - T_4 a w_2' \right) B e^{-at} = 0
\end{aligned}\right\} \tag{13}$$

Referring back to equations (10) we find that each one of the equations contains w_1 and w_2, and, in order that these equations may hold true for all values of time, it is evident that w_1 and w_2 must be similar functions of time; and therefore the ratio of $\dfrac{w_1}{w_2}$ or $\dfrac{dw_1}{dw_2}$ will be a constant. We can therefore choose such a value for a which will make the following relations obtain:

$$\left.\begin{aligned}
B\left\{ P_2 - 2P_1 a \right\} \frac{dw_2'}{dt} + \frac{A P_4}{k_3} \frac{dw_1'}{dt} = 0 \\
A\left\{ T_2 - 2T_1 a \right\} \frac{dw_1'}{dt} + B T_4 k_3 \frac{dw_2'}{dt} = 0
\end{aligned}\right\} \tag{14}$$

Eliminating $\dfrac{A\dfrac{dw_1'}{dt}}{B\dfrac{dw_2}{dt}}$ from the above equations we get:

$$\frac{(P_2 - 2P_1 a)k_3}{P_4} = \frac{T_4 k_3}{T_2 - 2T_1 a}$$

or

$$T_2 P_2 - 2(P_1 T_2 + T_1 P_2)a + 4T_1 P_1 a^2 = T_4 P_4$$

and

$$a = \frac{2(P_1 T_2 + T_1 P_2) \pm \sqrt{4(P_1 T_2 + T_1 P_2)^2 - 16T_2 P_2 T_1 P_1 + 16T_1 P_1 T_4 P_4}}{8P_1 T_1} \quad (15)$$

Since, however, $T_1 = -P_1$, equation (15) will reduce to the following:

$$a = \frac{(T_2 - P_2) \pm \sqrt{(T_2 + P_2)^2 - 4P_4 T_4}}{4T_1} \quad (16)$$

Since a has two distinct values, equations (12) will be written in the following form:

$$\left.\begin{array}{l} w_1 = \{A_1 e^{-a_1 t} + A_2 e^{-a_2 t}\}w_1' \\ w_2 = \{B_1 e^{-a_1 t} + B_2 e^{-a_2 t}\}w_2' \end{array}\right\} \quad (17)$$

From equations (13) and (14) we obtain the following two equations:

$$\left.\begin{array}{l} \left\{P_1 \dfrac{d^2 w_2'}{dt^2} + P_1 a^2 w_2' - P_2 a w_2' + P_3 w_2'\right\} B - \dfrac{aP_4}{k_3} A w_1' = 0 \\[4mm] \left\{T_1 \dfrac{d^2 w_1'}{dt^2} + T_1 a^2 w_1' - T_2 a w_1' + T_3 w_1'\right\} A - k_3 T_4 B a w_2' = 0 \end{array}\right\} \quad (18)$$

Put for brevity

$$\left.\begin{array}{l} P_1 a^2 - P_2 a + P_3 = N_1 \\ T_1 a^2 - T_2 a + T_3 = N_2 \end{array}\right\} \quad (19)$$

Then equations (18) will become:

$$\left.\begin{array}{l} \left\{P_1 \dfrac{d^2 w_2'}{dt^2} + N_1 w_2'\right\} B - A \dfrac{aP_4}{k_3} w_1' = 0 \\[4mm] \left\{T_1 \dfrac{d^2 w_1'}{dt^2} + N_2 w_1'\right\} A - Ba T_4 k_3 w_2' = 0 \end{array}\right\} \quad (20)$$

Let us assume the following solutions,

$$\left.\begin{array}{l} w_1' = De^{i\lambda t} \\ w_2' = Fe^{i\lambda t} \end{array}\right\} \tag{21}$$

Introducing these values in (20) we obtain:

$$\left.\begin{array}{l} \left\{-P_1\lambda^2 + N_1\right\}BF - \dfrac{aP_4}{k_3}AD = 0 \\ \left\{-T_1\lambda^2 + N_2\right\}AD - aT_4k_3BF = 0 \end{array}\right\} \tag{22}$$

Eliminating from these two equations $\dfrac{AD}{BF}$ we get

$$\frac{(-P_1\lambda^2 + N_1)k_3}{aP_4} = \frac{aT_4k_3}{-T_1\lambda^2 + N_2}$$

or

$$T_1P_1\lambda^4 - (N_1T_1 + N_2P_1)\lambda^2 + N_1N_2 - a^2P_4T_4 = 0$$

and

$$\lambda^2 = \frac{N_1T_1 + N_2P_1 \pm \sqrt{(N_1T_1 + N_2P_1)^2 - 4N_1N_2P_1T_1 + 4T_1P_1P_4T_4a^2}}{2T_1P_1}$$

Remembering that $T_1 = -P_1$, the above equation will reduce to,

$$\lambda^2 = \frac{N_2 - N_1 \pm \sqrt{(N_2 + N_1)^2 - 4P_4T_4a^2}}{2T_1} \tag{23}$$

Since λ has two distinct values, we shall have

$$w_1' = D_1e^{i\lambda_1 t} + D_2e^{i\lambda_2 t}$$

$$w_2 = F_1e^{i\lambda_1 t} + F_2e^{i\lambda_2 t}$$

and the complete solutions of w_1 and w_2 will therefore be

$$\left.\begin{array}{l} w_1 = \left\{A_1e^{-a_1 t} + A_2e^{-a_2 t}\right\}\left\{D_1e^{i\lambda_1 t} + D_2e^{i\lambda_2 t}\right\} \\ w_2 = \left\{B_1e^{-a_1 t} + B_2e^{-a_2 t}\right\}\left\{F_1e^{i\lambda_1 t} + F_2e^{i\lambda_2 t}\right\} \end{array}\right\} \tag{24}$$

Put now

$$A_1D_1 = K_1, \qquad A_1D_2 = K_2, \qquad A_2D_1 = K_3, \qquad A_2D_2 = K_4$$
$$B_1F_1 = K_5, \qquad B_1F_2 = K_6, \qquad B_2F_1 = K_7, \qquad B_2F_2 = K_8$$

Equations (24) will reduce to

$$\left. \begin{aligned} w_1 &= K_1 e^{-a_1 t} e^{i\lambda_1 t} + K_2 e^{-a_1 t} e^{i\lambda_2 t} + K_3 e^{-a_2 t} e^{i\lambda_1 t} + K_4 e^{-a_2 t} e^{i\lambda_2 t} \\ w_2 &= K_5 e^{-a_1 t} e^{i\lambda_1 t} + K_6 e^{-a_1 t} e^{i\lambda_2 t} + K_7 e^{-a_2 t} e^{i\lambda_1 t} + K_8 e^{-a_2 t} e^{i\lambda_2 t} \end{aligned} \right\} \qquad (25)$$

and by equation (4) we have,

$$\left. \begin{aligned} v_1 &= m_1 w_1 + m_2 w_2 \\ &= (m_1 K_1 + m_2 K_5)\, e^{-a_1 t} e^{i\lambda_1 t} + (m_1 K_2 + m_2 K_6) e^{-a_1 t} e^{i\lambda_2 t} + \\ &\qquad (m_1 K_3 + m_2 K_7)\, e^{-a_2 t} e^{i\lambda_1 t} + (m_1 K_4 + m_2 K_8)\, e^{-a_2 t} e^{i\lambda_2 t} \\[6pt] V_2 &= m_3 w_1 + m_4 w_2 \\ &= (m_3 K_1 + m_4 K_5)\, e^{-a_1 t} e^{i\lambda_1 t} + (m_3 K_2 + m_4 K_6)\, e^{-a_1 t} e^{i\lambda_2 t} + \\ &\qquad (m_3 K_3 + m_4 K_7) e^{-a_2 t} e^{i\lambda_1 t} + (m_3 K_4 + m_4 K_8) e^{-a_2 t} e^{i\lambda_2 t} \end{aligned} \right\} \qquad (26)$$

Equations (26) give the complete expressions for the potentials in the primary and secondary, and the expressions for the currents can be obtained, of course, from equations (26) by the aid of equations (2).

In the evaluations of the constants a_1, a_2, λ_1, and λ_2 it will be shown that they are all real quantities, and hence it is evident that a_1 and a_2 are the damping factors, and λ_1 and λ_2 are the frequency constants.

The values of a and λ are given by equations (16) and (23), respectively, their evaluation in terms of the constants of the circuits for the most general case is somewhat laborious, and I shall therefore consider only the most important case which is usually adopted in practice, that of resonance. In that case we have $a_2 = a_1$, and therefore by equation (8), we get

$$k_1 = -k_2 = \sqrt{\frac{d_1}{d_2}} \qquad (27)$$

Using these values in equations (16), we have

$$T_2 - P_2 = 2(a_2b_1 + a_1b_2)\sqrt{\frac{d_1}{d_2}}$$

$$T_2 + P_2 = -2d_1(b_1 + b_2)$$

$$P_4 T_4 = -\frac{d_1}{d_2}\Big\{(a_2b_1 - a_1b_2)^2 - d_1d_2(b_1 - b_2)^2\Big\}$$

and therefore

$$a = \frac{2(a_2b_1 + a_1b_2)\sqrt{\frac{d_1}{d_2}} \pm \sqrt{4d_1{}^2(b_1 + b_2)^2 + 4\frac{d_1}{d_2}\big\{(a_2b_1 - a_1b_2)^2 - d_1d_2(b_1 - b_2)^2\big\}}}{8(a_1a_2 - d_1d_2)\sqrt{\frac{d_1}{d_2}}}$$

$$= \frac{a_2b_1 + a_1b_2 \pm \sqrt{(a_2b_1 - a_1b_2)^2 + 4d_1d_2b_1b_2}}{4(a_1a_2 - d_1d_2)} \tag{28}$$

Substituting the values of the various constants and simplifying we get:

$$a = \frac{L_2R_1 + L_1R_2 \pm \sqrt{(L_2R_1 - L_1R_2)^2 + 4M^2R_1R_2}}{4(L_1L_2 - M^2)} \tag{29}$$

If we designate the natural damping factors of the primary and secondary by γ_1 and γ_2, respectively; that is, put $\frac{R_1}{2L_1} = \gamma_1$ and $\frac{R_2}{2L_2} = \gamma_2$, and also denote the coefficient of coupling $\frac{M^2}{L_1L_2}$ by μ^2, then equations (29) may be written in the following form:

$$a_1 = \frac{\gamma_1 + \gamma_2 + \sqrt{(\gamma_1 - \gamma_2)^2 + 4\gamma_1\gamma_2\mu^2}}{2(1 - \mu^2)}$$

$$a_2 = \frac{\gamma_1 + \gamma_2 - \sqrt{(\gamma_1 - \gamma_2)^2 + 4\gamma_1\gamma_2\mu^2}}{2(1 - \mu^2)} \tag{30}$$

When the coefficient of coupling is zero, then the above two equations will reduce to $a_1 = \gamma_1$ and $a_2 = \gamma_2$

That is, the damping factors will be the natural damping factors of the primary and secondary.

To obtain the expressions for λ_1 and λ_2 in terms of the constants of the circuits, we note that

$$N_2 - N_1 = -2P_1 a^2 + (P_2 - T_2)a + T_3 - P_3$$

$$= 4(a_1 a_2 - d_1 d_2)\,k_1 a^2 + 2(a_2 b_1 + a_1 b_2)\,k_1 a + 2(a_1 + a_2)\,k_1$$

$$N_2 + N_1 = -(P_2 + T_2)a + P_3 + T_3 = 2d_1(b_1 + b_2)a + 4d_1$$

$$P_4 T_4 = -\frac{d_1}{d_2}\left\{(a_2 b_1 - a_1 b_2)^2 + d_1 d_2(b_1 - b_2)^2\right\}$$

Introducing these values in equation (23) we obtain:

$$
\begin{aligned}
\lambda^2 &= \frac{4(a_1 a_2 - d_1 d_2)k_1 a^2 + 2(a_2 b_1 + a_1 b_2)k_1 a + 2(a_1 + a_2)k_1}{4(a_1 a_2 - d_1 d_2)k_1} \\[2mm]
&\quad \pm \frac{\sqrt{\left\{2d_1(b_1 + b_2)a - 4d_1\right\}^2 + \dfrac{4d_1}{d_2}\left\{(a_2 b_1 - a_1 b_2)^2 + d_1 d_2(b_1 - b_2)^2\right\}a^2}}{4(a_1 a_2 - d_1 d_2)k_1} \\[3mm]
&= \frac{2(a_1 a_2 - d_1 d_2)a^2 + (a_2 b_1 + a_1 b_2)a + (a_1 + a_2)}{2(a_1 a_2 - d_1 d_2)} \\[2mm]
&\quad \pm \frac{\sqrt{d_1 d_2\left\{(b_1 + b_2)a - 2\right\}^2 + \left\{(a_2 b_1 - a_1 b_2)^2 + d_1 d_2(b_1 - b_2)^2\right\}a^2}}{2(a_1 a_2 - d_1 d_2)}
\end{aligned}
$$

$$(31)$$

It is seen from equation (31) that all the terms in the expression of λ^2 are positive, and therefore when we use the positive sign before the radical; that is, the value of λ_1 will certainly be positive, but even when we use the negative sign before the radical— that is, the value of λ_2 will also be positive, because the principal term in the part of the expression outside the radical is $a_1 + a_2$, the magnitude of which is of the order LC, while the principal term under the radical is $4d_1 d_2$ and its magnitude is of the order MC, hence the expression of λ^2 as given by (31) will be always positive, and therefore λ is a real quantity. It is also seen from equation (28) that a is a real quantity, hence the a's as given by equation (28) are the damping factors, and the λ's as given by equation (31) are the frequency constants.

It may also be noted that in all cases which may arise in practice, the terms in equation (31) which contain a as a factor

are usually very small as compared with the other terms. If we neglect the resistance entirely, then equation (31) will reduce to the following:

$$\lambda^2 = \frac{a_1 + a_2 \pm \sqrt{4 d_1 d_2}}{2 (a_1 a_2 - d_1 d_2)} \tag{32}$$

which agrees, of course, with the results of Oberbeck [9] who developed the theory on the assumption of no resistance.

Having determined the damping factors, and the frequency constants, it remains yet to evaluate the constants of integration to obtain the complete solution of the problem.

The initial conditions are:

When $t = 0$, $v_1 = E$, $v_2 = 0$, $I_1 = 0$, $I_2 = 0$

The expressions for v_1 and v_2 are given by equation (26).

$$
\begin{aligned}
v_1 &= (m_1 K_1 + m_2 K_5) e^{-a_1 t} e^{i\lambda_1 t} + (m_1 K_2 + m_2 K_6) e^{-a_1 t} e^{i\lambda_2 t} + \\
&\quad (m_1 K_3 + m_2 K_7) e^{-a_2 t} e^{i\lambda_1 t} + (m_1 K_4 + m_2 K_8) e^{-a_2 t} e^{i\lambda_2 t} \\
v_2 &= (m_3 K_1 + m_4 K_5) e^{-a_1 t} e^{i\lambda_1 t} + (m_3 K_2 + m_4 K_6) e^{-a_1 t} e^{i\lambda_2 t} + \\
&\quad (m_3 K_3 + m_4 K_7) e^{-a_2 t} e^{i\lambda_1 t} + (m_3 K_4 + m_4 K_8) e^{-a_2 t} e^{i\lambda_2 t}
\end{aligned}
\tag{26 bis}
$$

In order that the values of w_1 and w_2 as given by equations (25) should satisfy equations (22) there must exist a certain relation between the various K's which enter in equations (25). If we introduce the values of w_1 and w_2 as given by equations (25) into (22) we find that the relation between the constants are as follows:

$$
\left.
\begin{aligned}
\frac{K_5}{K_1} &= \frac{-T_1 \lambda_1^2 + N_2}{a_1 T_4 k_3} \quad \text{or} \quad K_5 = \frac{-T_1 \lambda_1^2 + N_2}{a_1 T_4 k_3} K_1 = \frac{Q_1}{k_3} K_1 \\
\frac{K_6}{K_2} &= \frac{-T_1 \lambda_2^2 + N_2}{a_1 T_4 k_3} \quad \text{or} \quad K_6 = \frac{-T_1 \lambda_2^2 + N_2}{a_1 T_4 k_3} K_2 = \frac{Q_2}{k_3} K_2 \\
\frac{K_7}{K_3} &= \frac{-T_1 \lambda_1^2 + N_2}{a_2 T_4 k_3} \quad \text{or} \quad K_7 = \frac{-T_1 \lambda_1^2 + N_2}{a_2 T_4 k_3} K_3 = \frac{Q_3}{k_3} K_3 \\
\frac{K_8}{K_4} &= \frac{-T_1 \lambda_2^2 + N_2}{a_2 T_4 k_3} \quad \text{or} \quad K_8 = \frac{-T_1 \lambda_2^2 + N_2}{a_2 T_4 k_3} K_4 = \frac{Q_4}{k_3} K_4
\end{aligned}
\right\}
\tag{33}
$$

[9] A. Oberbeck, Wied. Ann. **55**, p. 623; 1895.

Introducing these values in equation (26) and remembering that $k_3 = \dfrac{m_4}{m_1}$, we get:

$$
\left.
\begin{aligned}
v_1 &= m_1 K_1 (1 + k_2 Q_1) e^{-a_1 t} e^{i\lambda_1 t} + m_1 K_2 (1 + k_2 Q_2) e^{-a_1 t} e^{i\lambda_2 t} + \\
&\quad m_1 K_3 (1 + k_2 Q_3) e^{-a_2 t} e^{i\lambda_1 t} + m_1 K_4 (1 + k_2 Q_4) e^{-a_2 t} e^{i\lambda_2 t} \\
v_2 &= m_3 K_1 (1 + k_1 Q_1) e^{-a_1 t} e^{i\lambda_1 t} + m_3 K_2 (1 + k_1 Q_2) e^{-a_1 t} e^{i\lambda_2 t} + \\
&\quad m_3 K_3 (1 + k_1 Q_3) e^{-a_2 t} e^{i\lambda_1 t} + m_3 K_4 (1 + k_1 Q_4) e^{-a_2 t} e^{i\lambda_2 t}
\end{aligned}
\right\} \quad (34)
$$

The expressions for the currents in the primary and secondary are as follows:

$$
\left.
\begin{aligned}
I_1 &= c_1 m_1 \{ K_1 (1 + k_2 Q_1)(i\lambda_1 - a_1) e^{-a_1 t} e^{i\lambda_1 t} + \\
&\quad K_2 (1 + k_2 Q_2)(i\lambda_2 - a_1) e_1^{-a_1 t} e^{i\lambda_2 t} + \\
&\quad K_3 (1 + k_2 Q_3)(i\lambda_1 - a_2) e^{-a_2 t} e^{i\lambda_1 t} + \\
&\quad K_4 (1 + k_2 Q_4)(i\lambda_2 - a_2) e^{-a_2 t} e^{i\lambda_2 t} \} \\
I_2 &= c_2 m_3 \{ K_1 (1 + k_1 Q_1)(i\lambda_1 - a_1) e^{-a_1 t} e^{i\lambda_1 t} + \\
&\quad K_2 (1 + k_1 Q_2)(i\lambda_2 - a_1) e^{-a_1 t} e^{i\lambda_1 t} + \\
&\quad K_3 (1 + k_1 Q_3)(i\lambda_1 - a_2) e^{-a_2 t} e^{i\lambda_1 t} + \\
&\quad K_4 (1 + k_1 Q_4)(i\lambda_2 - a_2) e^{-a_2 t} e^{i\lambda_2 t} \}
\end{aligned}
\right\} \quad (35)
$$

When $t = 0$, we have:

$$
\left.
\begin{aligned}
&m_1 \{ K_1 (1 + k_2 Q_1) + K_2 (1 + k_2 Q_2) + K_3 (1 + k_2 Q_3) + \\
&\qquad K_4 (1 + k_2 Q_4) \} = E \\[4pt]
&m_3 \{ K_1 (1 + k_1 Q_1) + K_2 (1 + k_1 Q_2) + K_3 (1 + k_1 Q_3) + \\
&\qquad K_4 (1 + k_1 Q_4) \} = 0 \\[4pt]
&c_1 m_1 \{ K_1 (1 + k_2 Q_1)(i\lambda_1 - a_1) + K_2 (1 + k_2 Q_2)(i\lambda_2 - a_1) + \\
&\qquad K_3 (1 + k_2 Q_3)(i\lambda_1 - a_2) + K_4 (1 + k_2 Q_4)(i\lambda_2 - a_2) \} = 0 \\[4pt]
&c_2 m_3 \{ K_1 (1 + k_1 Q_1)(i\lambda_1 - a_1) + K_2 (1 + k_1 Q_2)(i\lambda_2 - a_1) + \\
&\qquad K_3 (1 + k_1 Q_3)(i\lambda_1 - a_2) + K_4 (1 + k_1 Q_4)(i\lambda_2 - a_2) \} = 0
\end{aligned}
\right\} \quad (36)
$$

We have here four equations from which we can by the use of determinates obtain the values of the four constants K_1, K_2, K_3, and K_4. The results, however, as thus obtained will be extremely complicated, and it will therefore be advisable to get approximate values of the constants if we will hereby simplify matters materially. As a matter of fact, the approximations that we will adopt here will be very close to the absolute values, and, furthermore, in our final results it is the ratio of the secondary potential to the primary potential that we seek to determine, and since these constants enter in the expression of v_1 and v_2, hence the approximation will affect the numerator and denominator alike, and the ratio will be a still closer approximation to the absolute value.

In examining equations (35) we find that each term of the right-hand sides contain a factor $(i\lambda - a)$. Now, in practice, a is generally very small compared with λ, but it must be further noted that if we take the real part (35), the term $i\lambda - a$ will become $\sqrt{\lambda^2 + a^2}$, and a^2 is generally negligible compared with λ^2. Hence we may neglect a compared with $i\lambda$, and under these conditions equations (36) will reduce to the following:

$$\left.\begin{array}{l} \{K_1(1 + k_2Q_1) + K_3(1 + k_2Q_3)\} + \\[4pt] \quad \{K_2(1 + k_2Q_2) + K_4(1 + k_2Q_4)\} = \dfrac{E}{m_1} \\[10pt] i\lambda_1\{K_1(1 + k_2Q_1) + K_3(1 + k_2Q_3)\} + \\[4pt] \quad i\lambda_2\{K_2(1 + k_2Q_2) + K_4(1 + k_2Q_4)\} = 0 \\[10pt] \{K_1(1 + k_1Q_1) + K_3(1 + k_1Q_3)\} + \\[4pt] \quad \{K_2(1 + k_1Q_2) + K_4(1 + k_1Q_4)\} = 0 \\[10pt] i\lambda_1\{K_1(1 + k_1Q_1) + K_3(1 + k_1Q_3)\} + \\[4pt] \quad i\lambda_2\{K_2(1 + k_1Q_2) + K_4(1 + k_1Q_4)\} = 0 \end{array}\right\} \quad (37)$$

In order that the last two equations of (37) be both satisfied, we must have each one of the bracket expressions separately equal to zero; that is, we have the following relations:

$$K_1(1 + k_1Q_1) + K_3(1 + k_1Q_3) = 0$$
$$K_2(1 + {}_1kQ_2) + K_4(1 + k_1Q_4) = 0$$

and therefore,

$$\frac{K_1}{K_3} = -\frac{1 + k_1Q_3}{1 + k_1Q_1} \quad \frac{K_2}{K_4} = -\frac{1 + k_1Q_4}{1 + k_1Q_2} \tag{38}$$

From the first two equations of (37) we get:

$$\left. \begin{aligned} K_2(1 + k_2Q_2) + K_4(1 + k_2Q_4) &= \frac{E}{m_1} \frac{\lambda_1}{\lambda_1 - \lambda_2} \\[2ex] K_1(1 + k_2Q_1) + K_3(1 + k_2Q_3) &= \frac{E}{m_1} \frac{\lambda_2}{\lambda_2 - \lambda_1} \end{aligned} \right\} \tag{39}$$

From (38) and (39) we can determine the values of the constants, which are as follows:

$$\left. \begin{aligned} K_1 &= \frac{\dfrac{E}{m_1}\lambda_2(1 + k_1Q_3)}{(\lambda_2 - \lambda_1)\{(1 + k_2Q_1)(1 + k_1Q_3) - (1 + k_2Q_3)(1 + k_1Q_2)\}} \\[1ex] &= \frac{\dfrac{E}{m_1}\lambda_2(1 + k_1Q_3)}{S_1} \\[3ex] K_2 &= \frac{\dfrac{E}{m_1}\lambda_1(1 + k_1Q_4)}{(\lambda_1 - \lambda_2)\{(1 + k_2Q_2)(1 + k_1Q_4) - (1 + k_2Q_4)(1 + k_1Q_2)\}} \\[1ex] &= \frac{\dfrac{E}{m_1}\lambda_1(1 + k_1Q_4)}{S_2} \\[3ex] K_3 &= -\frac{\dfrac{E}{m_1}\lambda_2(1 + k_1Q_1)}{S_1} \quad K_4 = -\frac{\dfrac{E}{m_1}\lambda_1(1 + k_1Q_2)}{S_2} \end{aligned} \right\} \tag{41}$$

Introducing the values of the constants as given by equation (41) into equations (34) and (35), we shall have the complete expressions for the potentials and currents in the primary and secondary circuits, which are as follows:

$$
\begin{aligned}
v_1 = {}& \frac{E\lambda_2}{S_1}(1 + k_2 Q_1)(1 + k_1 Q_3)e^{-a_1 t}e^{i\lambda_1 t} + \\
& \frac{E\lambda_1}{S_2}(1 + k_1 Q_4)(1 + k_1 Q_2)e^{-a_1 t}e^{i\lambda_2 t} - \\[6pt]
& \frac{E\lambda_2}{S_1}(1 + k_1 Q_1)(1 + k_2 Q_3)e^{-a_2 t}e^{i\lambda_1 t} - \\
& \frac{E\lambda_1}{S_2}(1 + k_1 Q_2)(1 + k_2 Q_4)e^{-a_2 t}e^{i\lambda_2 t}
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
v_2 = {}& \frac{E\lambda_2}{k_1 S_1}(1 + k_1 Q_3)(1 + k_1 Q_1)e^{-a_1 t}e^{i\lambda_1 t} + \\
& \frac{E\lambda_1}{k_1 S_2}(1 + k_1 Q_4)(1 + k_1 Q_2)e^{-a_1 t}e^{i\lambda_2 t} - \\[6pt]
& \frac{E\lambda_2}{k_1 S_1}(1 + k_1 Q_1)(1 + k_1 Q_3)e^{-a_2 t}e^{i\lambda_1 t} - \\
& \frac{E\lambda_1}{k_1 S_2}(1 + k_1 Q_2)(1 + k_1 Q_4)e^{-a_2 t}e^{i\lambda_2 t}
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
I_1 = {}& \frac{C_1 E\lambda_2}{S_1}(1 + k_2 Q_1)(1 + k_1 Q_3)i\lambda_1 e^{-a_1 t}e^{i\lambda_1 t} + \\
& \frac{C_1 E\lambda_1}{S_2}(1 + k_2 Q_2)(1 + k_1 Q_4)i\lambda_2 e^{-a_1 t}e^{i\lambda_2 t} - \\[6pt]
& \frac{C_1 E\lambda_2}{S_1}(1 + k_2 Q_3)(1 + k_1 Q_1)i\lambda_1 e^{-a_2 t}e^{i\lambda_1 t} - \\
& \frac{C_1 E\lambda_1}{S_2}(1 + k_2 Q_4)(1 + k_1 Q_2)i\lambda_2 e^{-a_2 t}e^{i\lambda_2 t}
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
I_2 = {}& \frac{C_2 E\lambda_2}{k_1 S_1}(1 + k_1 Q_1)(1 + k_1 Q_3)i\lambda_1 e^{-a_1 t}e^{i\lambda_1 t} + \\
& \frac{C_2 E\lambda_1}{k_1 S_2}(1 + k_1 Q_2)(1 + k_1 Q_4)i\lambda_2 e^{-a_1 t}e^{i\lambda_2 t} - \\[6pt]
& \frac{C_2 E\lambda_2}{k_1 S_1}(1 + k_1 Q_3)(1 + k_1 Q_1)i\lambda_1 e^{-a_2 t}e^{i\lambda_1 t} - \\
& \frac{C_2 E\lambda_1}{k_1 S_2}(1 + k_1 Q_4)(1 + k_1 Q_2)i\lambda_2 e^{-a_2 t}e^{i\lambda_2 t}
\end{aligned}
\tag{45}
$$

Let us now use the following abbreviations:

$$\frac{E\lambda_2(1+k_2Q_1)(1+k_1Q_3)}{S_1}=H_1, \qquad \frac{E\lambda_1(1+k_1Q_2)(1+k_2Q_4)}{S_2}=H_4 \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\end{array}}\right\}$$

$$\frac{E\lambda_1(1+k_1Q_4)(1+k_2Q_2)}{S_2}=H_2, \qquad \frac{E\lambda_2(1+k_1Q_3)(1+k_1Q_1)}{k_1S_1}=H_5 \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} (46)$$

$$\frac{E\lambda_2(1+k_1Q_1)(1+k_1Q_3)}{S_1}=H_3, \qquad \frac{E\lambda_1(1+k_1Q_4)(1+k_1Q_5)}{k_1S_2}=H_6 \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\}$$

Then the real parts of equations (42)–(45) may be written in the following form:

$$v_1=H_1e^{-a_1t}\cos\lambda_1t+H_2e^{-a_1t}\cos\lambda_2t-H_3e^{-a_2t}\cos\lambda_1t-H_4e^{-a_2t}\cos\lambda_2t \;(47)$$

$$v_2=H_5e^{-a_1t}\cos\lambda_1t+H_6e^{-a_1t}\cos\lambda_2t-H_5e^{-a_2t}\cos\lambda_1t-H_6e^{-a_2t}\cos\lambda_2t \;(48)$$

$$\left.\begin{array}{c}I_1=C_1H_1\lambda_1e^{-a_1t}\sin\lambda_1t+C_1H_2\lambda_2e^{-a_1t}\sin\lambda_2t-C_1H_3\lambda_1e^{-a_2t}\sin\lambda_1t-\\ C_1H_4\lambda_2e^{-a_2t}\sin\lambda_2t\end{array}\right\} (49)$$

$$\left.\begin{array}{c}I_2=C_2H_5\lambda_1e^{-a_1t}\sin\lambda_1t+C_2H_6\lambda_2e^{-a_1t}\sin\lambda_2t-C_2H_5\lambda_1e^{-a_2t}\sin\lambda_1t-\\ C_2H_6\lambda_2e^{-a_2t}\sin\lambda_2t\end{array}\right\} (50)$$

or we may write the above equations in the following form:

$$\left.\begin{array}{l}v_1=\{H_1e^{-a_1t}-H_3e^{-a_2t}\}\cos\lambda_1t+\{H_2e^{-a_1t}-H_4e^{-a_2t}\}\cos\lambda_2t\\[4pt] v_2=H_5\{e^{-a_1t}-e^{-a_2t}\}\cos\lambda_1t+H_6\{e^{-a_1t}-e^{-a_2t}\}\cos\lambda_2t\\[4pt] I_1=C_1\lambda_1\{H_1e^{-a_1t}-H_3e^{-a_2t}\}\sin\lambda_1t+C_1\lambda_2\{H_2e^{-a_1t}-H_4e^{-a_2t}\}\sin\lambda_2t\\[4pt] I_2=C_2\lambda_1H_5\{e^{-a_1t}-e^{-a_2t}\}\sin\lambda_1t+C_2\lambda_2H_6\{e^{-a_1t}-e^{-a_2t}\}\sin\lambda_2t\end{array}\right\} (51)$$

Equations (52) give the complete solution to the problem of electromagnetic coupled systems. It is seen that the potential and current in the primary and secondary circuits consist of two distinct waves, each one of which has a different periodicity, and each wave has two different damping factors. It is also evident that each wave in the primary circuit will produce a corresponding wave in the secondary circuit.

We may thus write:

$$\left.\begin{array}{c}v_1=v_1'+v_1''\\[6pt] v_2=v_2'+v_2''\end{array}\right\} (52)$$

Where v_1' and v_2' correspond to the terms of frequency λ_1, and v_1'', v_2'' correspond to the terms of frequency λ_2.

If we should introduce the values of the electrical constants that are generally used in practice in equations (33), it will be found that the constants Q_1, Q_2, Q_3, and Q_4 are large quantities, their magnitude being of the order of one hundred or more. If we neglect unity compared with Q, we shall find by referring to equations (46) that $H_1 = H_3$ and $H_2 = H_4$.

The first and third equations of (51) will therefore reduce to,

$$v_1 = H_1(e^{-a_1 t} - e^{-a_2 t}) \cos \lambda_1 t + H_2(e^{-a_1 t} - e^{-a_2 t}) \cos \lambda_2 t$$

$$I_1 = C_1 \lambda_1 H_1(e^{-a_1 t} - e^{-a_2 t}) \cos \lambda_1 t + C_1 \lambda_2 H_2(e^{-a_1 t} - e^{-a_2 t}) \cos \lambda_2 t \tag{53}$$

If we assume that there is no resistance in the circuits; that is, $a_1 = a_2 = 0$, then the values of the various Q's as given by equations (33) will be infinite, and we may therefore neglect unity as compared with kQ in equations (46). Under these conditions we find that,

$$\frac{v_2'}{v_1'} = \frac{v_2''}{v_1''} = \frac{1}{k_1} = \sqrt{\frac{C_1}{C_2}} \tag{54}$$

which agrees with the result obtained in the development of the simple theory on the assumption of no resistance in the circuits.

It will perhaps be of some interest to work out one or two examples so as to get an idea of the relative magnitude of the various constants which enter in the final equations.

Let us assume the following values for the resistances, inductance, and capacity of the primary and secondary circuits:

$R_1 = 1$ ohm $= 10^9$ cm, $\qquad L_1 = 6200$ cm, $\qquad C_1 = 5300 \times 10^{-21}$

$R_2 = 50$ ohm $= 50 \times 10^9$ cm, $\qquad L_2 = 73000$ cm, $\qquad C_2 = 450 \times 10^{-21}$

and let us also consider two different cases of different degrees of coupling $\mu = 0.1$ and $\mu = 0.4$. Then we shall have the following values for our various constants (see Table I).

From the values given in the table, it is seen that the differences between H_1 and H_3, H_2 and H_4, are small, so that our assumption in equation (53) that $H_1 = H_3$ is justifiable. We also

TABLE I.

Calculated Values of Constants for Two Degrees of Coupling.

μ	$L_1C_1=L_2C_2$	λ_1	λ_2	α_1	α_2	Q_1	Q_2	Q_3	Q_4	S_1	S_2
0.1	33×10^{-15}	5.5×10^6	5.1×10^6	38×10^4	9×10^4	-91	161	-375	662	—	120×10^6
0.4	33×10^{-15}	7.1×10^6	4.65×10^6	20×10^4	5.4×10^4	-3461	1440	-11815	5369	-12280×10^6	5775×10^6

μ	H_1	H_2	H_3	H_4	H_5	H_6	$\dfrac{H_5}{H_1}$	$\dfrac{H_6}{H_2}$	$\sqrt{\dfrac{C_1}{C_2}}$
0.1	192	-433	218	-444	-733	1503	3.8	3.5	3.42
0.4	1530	-662	1527	-664	-5090	2215	3.33	3.34	

see that, in the case of loose coupling ($\mu = 0.1$), the frequencies do not differ much from each other, and the effect will be approximately the same as if we had an oscillation of one frequency. In the case, however, of strong coupling ($\mu = 0.4$) the frequency constants differ materially from each other, so in this case we must consider the effect as that due to two distinct oscillations of different frequencies and different amplitudes.

DIRECT COUPLED CIRCUITS.

The method that was used in the discussion of the problem of electromagnetically coupled circuits is in general applicable to the discussion of the problem of direct coupled circuits. The treatment of this problem is so closely analogous to that of the preceding case that it may appear almost like a repetition, yet the results obtained, we trust, justify the additional labor involved in the development of this problem. As was said in the introduction, no adequate treatment of this problem has, so far as we know, ever been given; in fact, not even an approximate determination of the damping factors has ever been attempted, and it is hoped therefore that the results obtained will be of considerable interest.

In the discussion of this problem we shall use similar symbols, as in the previous case, but they may have the same or different meanings attached to them, and therefore in reading this part of the paper it must be remembered that it is a distinct and separate part, and all the symbols used will be defined in their proper place.

Let us denote by I_1 the current in the primary coil, I_2 the current in the antennæ, and I_3 the current in the condenser of the primary circuit; let V_1 denote the potential across the condensers, and V_2 the potential of the antenna. If L_1, R_1, C_1 and L_2, R_2, C_2 are the inductance, resistance, and capacity of the primary circuit and antenna, respectively, we shall have the following relations:

$$L_1 \frac{dI_1}{dt} + R_1 I_1 = V_1$$

$$L_2 \frac{dI_2}{dt} + R_2 I_2 + V_2 = V_1 \tag{55}$$

$$I_3 = C_1 \frac{dV_1}{dt}, \quad I_2 = C_2 \frac{dV_2}{dt}, \quad I_1 + I_2 + I_3 = 0$$

From the last relation given in equation (55) we have

$$I_1 = -(I_2 + I_3) = -C_2 \frac{dV_2}{dt} - C_3 \frac{dV_1}{dt}$$

and hence we can easily obtain the following two equations:

$$L_1 C_1 \frac{d^2 V_2}{dt^2} + L_1 C_1 \frac{d^2 V_1}{dt^2} + R_1 C_2 \frac{dV_2}{dt} + R_1 C_1 \frac{dV_1}{dt} + V_1 = 0$$

$$L_2 C_2 \frac{d^2 V_2}{dt^2} + R_2 C_2 \frac{dV_2}{dt} + V_2 - V_1 = 0 \tag{56}$$

or we may write the above two equations in this form:

$$a_1 \frac{d^2 V_1}{dt^2} + b_1 \frac{dV_1}{dt} + d_1 \frac{d^2 V_2}{dt^2} + h_1 \frac{dV_2}{dt} + V_1 = 0$$

$$a_2 \frac{d^2 V_2}{dt^2} + b_2 \frac{dV_2}{dt} + V_2 - V_1 = 0 \tag{57}$$

where

$$\begin{aligned} a_1 &= L_1 C_1, & b_1 &= R_1 C_1, & d_1 &= L_1 C_2, & h_1 &= R_1 C_2 \\ a_2 &= L_2 C_2, & b_2 &= R_2 C_2 \end{aligned} \tag{58}$$

As in the previous case we will assume,

$$\begin{aligned} V_1 &= m_1 w_1 + m_2 w_2 \\ V_2 &= m_3 w_1 + m_4 w_2 \end{aligned} \tag{59}$$

where m_1, m_2, m_3, and m_4 are arbitrary constants, and w_1 and w_2 are functions of t.

Introducing these values in (57) we obtain

$$
\left.
\begin{aligned}
&(a_1 m_1 + d_1 m_3)\frac{d^2 w_1}{dt^2} + (a_1 m_2 + d_1 m_4)\frac{d^2 w_2}{dt^2} + (b_1 m_1 + h_1 m_3)\frac{d w_1}{dt} + \\
&\qquad (b_1 m_2 + h_1 m_4)\frac{d w_2}{dt} + m_1 w_1 + m_2 w_2 = 0 \\[8pt]
&a_2 m_3 \frac{d^2 w_1}{dt^2} + a_2 m_4 \frac{d^2 w_2}{dt^2} + b_2 m_3 \frac{d w_1}{dt} + b_2 m_4 \frac{d w_2}{dt} + (m_3 - m_1)\,w_1 + \\
&\qquad (m_4 - m_2)\,w_2 = 0
\end{aligned}
\right\} \quad (60)
$$

Eliminating first w_1 and then w_2 we get the following two equations:

$$
\left.
\begin{aligned}
&\left\{(m_3 - m_1)(a_1 m_1 + d_1 m_3) - a_2 m_1 m_3\right\}\frac{d^2 w_1}{dt^2} + \\
&\left\{(m_3 - m_1)(a_1 m_2 + d_1 m_4) - a_2 m_1 m_4\right\}\frac{d^2 w_2}{dt^2} \\[4pt]
&+\left\{(m_3 - m_1)(b_1 m_1 + h_1 m_3) - b_2 m_1 m_3\right\}\frac{d w_1}{dt} + \\
&\left\{(m_3 - m_1)(b_1 m_2 + h_1 m_4) - b_2 m_1 m_4\right\}\frac{d w_2}{dt} + \\
&\left\{m_2(m_3 - m_1) - m_1(m_4 - m_2)\right\}w_2 = 0 \\[10pt]
&\left\{(m_4 - m_2)(a_1 m_1 + d_1 m_3) - a_2 m_2 m_3\right\}\frac{d^2 w_1}{dt^2} + \\
&\left\{(m_4 - m_2)(a_1 m_2 + d_1 m_4) - a_2 m_2 m_4\right\}\frac{d^2 w_2}{dt^2} \\[4pt]
&+\left\{(m_4 - m_2)(b_1 m_1 + h_1 m_3) - b_2 m_2 m_3\right\}\frac{d w_1}{dt} + \\
&\left\{(m_4 - m_2)(b_1 m_2 + h_1 m_4) - b_2 m_2 m_4\right\}\frac{d w_2}{dt} + \\
&\left\{m_1(m_4 - m_2) - m_2(m_3 - m_1)\right\}w_1 = 0
\end{aligned}
\right\} \quad (61)
$$

Let us now choose the constants m_1, m_2, m_3, and m_4 so as to make in the first equation of (61) the coefficient of $\dfrac{d^2 w_1}{dt^2}$ equal to zero, and in the second equation the coefficient of $\dfrac{d^2 w_2}{dt^2}$ equal to zero.

That is, we shall put

$$a_1 m_1 m_3 + d_1 m_3^2 - a_1 m_1^2 - d_1 m_1 m_3 - a_2 m_1 m_3 = 0$$

$$a_1 m_2 m_4 + d_1 m_4^2 - a_1 m_2^2 - d_1 m_2 m_4 - a_2 m_2 m_4 = 0$$

From which we obtain

$$\frac{m_1}{m_3} = \frac{m_2}{m_4} = \frac{a_1 - a_2 - d_1 \pm \sqrt{(a_1 - a_2 - d_1)^2 + 4a_1 d_1}}{2a_1}.$$

Using different signs before the radical for the two different ratios we get,

$$\left.\begin{aligned} \frac{m_1}{m_3} = k_1 &= \frac{a_1 - a_2 - d_1 + \sqrt{(a_1 - a_2 - d_1)^2 + 4a_1 d_1}}{2a_1} \\[2em] \frac{m_2}{m_4} = k_2 &= \frac{a_1 - a_2 - d_1 - \sqrt{(a_1 - a_2 - d_1)^2 + 4a_1 d_1}}{2a_1} \end{aligned}\right\} \quad (62)$$

To simplify matters we shall consider in this problem only the case when the two circuits are in resonance. This is in fact the most important case, which usually occurs in practice. Under these conditions we have

$$L_1 C_1 = (L_1 + L_2) C_2 \qquad (63)$$

or in our notation

$$a_1 = a_2 + d_1$$

and therefore equations (62) will reduce to

$$k_1 = \sqrt{\frac{d_1}{a_1}}, \qquad k_2 = -\sqrt{\frac{d_1}{a_1}} \qquad (64)$$

Expanding now the various coefficients in equations (61) and dividing through by $m_3 m_4$ we get

$$\left.\begin{aligned} &\left\{a_1 k_2 + d_1 - a_1 k_1 k_2 - d_1 k_1 - a_2 k_1\right\}\frac{d^2 w_2}{dt^2} + \\[1em] &\left\{b_1 k_2 - b_1 k_1 k_2 + h_1 - h_1 k_1 - b_2 k_1\right\}\frac{dw_2}{dt} \\[1em] &+\left\{k_2 - k_1\right\}w_2 + \frac{m_1}{m_4}\left\{b_1 - b_1 k_1 + \frac{h_1}{k_1} - h_1 - b_2\right\}\frac{dw_1}{dt} = 0 \end{aligned}\right\} \quad (65)$$

$$\left.\begin{aligned}
&\left\{a_1 k_1 - a_1 k_1 k_2 + d_1 - d_1 k_2 - a_2 k_2\right\}\frac{d^2 w_1}{dt^2} + \\
&\left\{b_1 k_1 - b_1 k_1 k_2 + h_1 - h_1 k_2 - b_2 k_2\right\}\frac{dw_1}{dt} \\
&+\left\{k_1 - k_2\right\}w_1 + \frac{m_2}{m_3}\left\{b_1 - b_1 k_2 + \frac{h_1}{k_2} - h_1 - b_2\right\}\frac{dw_2}{dt} = 0
\end{aligned}\right\} \quad (65)$$

Put for brevity $\dfrac{m_1}{m_4} = k_3$ and $\dfrac{m_2}{m_3} = \dfrac{m_4}{m_1}\dfrac{m_1}{m_4}\dfrac{m_2}{m_3} = \dfrac{k_1 k_2}{k_3}$ and then make the following abbreviations:

$$\left.\begin{aligned}
P_1 &= (a_1 + a_2)k_2 + 2d_1 - d_1 k_1 & T_1 &= (a_1 + a_2)k_1 + 2d_1 - d_1 k_2 \\
P_2 &= (b_1 + b_2 + h_1)k_2 - b_1 k_1 k_2 + h_1 & T_2 &= (b_1 + b_2 + h_1)k_1 - b_1 k_1 k_2 + h_1 \\
P_3 &= 2k_2 & T_3 &= 2k_1 \\
P_4 &= b_1 - h_1 - b_2 - b_1 k_1 + \frac{h_1}{k_1} & T_4 &= k_1 k_2\left\{b_1 - h_1 - b_2 - b_1 k_2 + \frac{h_1}{k_2}\right\}
\end{aligned}\right\} (66)$$

We may then write equations (65) in the following form:

$$\left.\begin{aligned}
P_1 \frac{d^2 w_2}{dt^2} + P_2 \frac{dw_2}{dt} + P_3 w_2 + k_3 P_4 \frac{dw_1}{dt} &= 0 \\
T_1 \frac{d^2 w_1}{dt^2} + T_2 \frac{dw_1}{dt} + T_3 w_1 + \frac{T_4}{k_3}\frac{dw_2}{dt} &= 0
\end{aligned}\right\} \quad (67)$$

Comparing equations (67) with equations (10) we see that they are of identical forms except that the constants in the two cases have different values. It is evident therefore that the solutions must also be of the same form. It is not necessary to repeat the work. We can write down at once the values of w_1 and w_2, which are given by equations (25),

$$\left.\begin{aligned}
w_1 &= K_1 e^{-a_1 t}e^{i\lambda_1 t} + K_2 e^{-a_1 t}e^{i\lambda_2 t} + K_3 e^{-a_2 t}e^{i\lambda_1 t} + K_4 e^{-a_2 t}e^{i\lambda_2 t} \\
w_2 &= K_5 e^{-a_1 t}e^{i\lambda_1 t} + K_6 e^{-a_1 t}e^{i\lambda_2 t} + K_7 e^{-a_2 t}e^{i\lambda_1 t} + K_8 e^{-a_2 t}e^{i\lambda_2 t}
\end{aligned}\right\} \quad (68)$$

Of course, the various constants which enter in equations (68) have entirely different values from the corresponding constants in

equations (25), and the most important point is the evaluation of these constants in terms of the electrical constants of the circuits.

The expressions for a and λ are, of course, of the same form as given by equations (15) and (23); that is,

$$a = \frac{P_1 T_2 + T_1 P_2 \pm \sqrt{(P_1 T_2 - P_2 T_1)^2 + 4P_1 T_1 P_4 T_4}}{4P_1 T_1} \tag{69}$$

$$\lambda^2 = \frac{N_1 T_1 + N_2 P_1 \pm \sqrt{(N_1 T_1 - N_2 P_1)^2 + 4P_1 T_1 P_4 T a^2}}{2T_1 P_1} \tag{70}$$

$$N_1 = P_1 a^2 - P_2 a + P_3 \qquad N_2 = T_1 a^2 - T_2 a + T_3$$

The values of the various constants are given by equations (66), and introducing their values we shall have

$$P_1 T_2 + T_1 P_2 = -\frac{2d_1}{a_1}\left\{(a_1 + a_2)(b_1 + b_2 + h_1) + d_1(b_2 - b_1 + h_1) - 2h_1 a_1\right\}$$

$$P_1 T_2 - T_1 P_2 = 4d_1(b_1 + b_2 + h_1)k_1 + 2b_1(a_1 + a_2)\frac{d_1}{a_1}k_2 - 2d_1 b_1\frac{d_1}{a_1}k_1 +$$
$$2h_1(a_1 + a_2) - 2h_1 d_1 k_1$$

$$P_4 T_4 = -\frac{d_1}{a_1}\left\{(b_1 - b_2)^2 - b_1^2 k_1^2\right\}$$

$$P_1 T_1 = 4d_1(d_1 - a_1)$$

now $\quad h_1 = R_1 C_2, \qquad b_1^2 k_1^2 = R_1^2 C_1 C_2, \qquad \frac{b_1}{a_1}d_1 = R_1 C_2 = h_1$

and generally in practice R_1 and C_2 are each comparatively very small quantities, so that we may neglect all the terms containing h_1 as a factor, and also the term $b_1^2 k_1^2$ and the term containing $\frac{b_1}{a_1}d_1$. Under these conditions we get

$$\left. \begin{aligned} P_1 T_2 + P_2 T_1 &= -\frac{2d_1}{a_1}\left\{(a_1 + a_2)(b_2 + b_1) - d_1(b_1 - b_2)\right\} \\[2mm] P_1 T_2 - T_1 P_2 &= 2d_1\left\{2b_2 + b_1 - b_1\frac{a_2}{a_1}\right\}k_1 \\[2mm] P_4 T_4 &= -\frac{d_1}{a_1}(b_1 - b_2)^2, \qquad P_1 T_1 = 4d_1(d_1 - a_1) \end{aligned} \right\} \tag{71}$$

Introducing these values in (69) we obtain

$$a = -\frac{2d_1}{a_1}\Big\{(a_1+a_2)(b_2+b_1) - d_1(b_1-b_2)\Big\} \pm \left. \frac{\sqrt{\frac{4d_1^3}{a_1}\left(2b_2+b_1-b_1\frac{a_2}{a_1}\right)^2 - \frac{16d_1^2}{a_1}(d_1-a_1)(b_1-b_2)^2}}{16d_1(d_1-a_1)} \right\} \quad (72)$$

If we expand the terms under the radical, and neglect terms containing b_1^2 as being very small compared with the terms containing b_2^2, we get,

$$a = \frac{(a_1+a_2)(b_1+b_2) - d_1(b_1-b_2) \pm \sqrt{4a_1^2b_2^2 - 4b_1b_2(a_2d_1 + a_1d_1 - 2a_1^2)}}{8(a_1^2 - a_1d_1)} \quad (73)$$

Substituting the values of a, b, and d as given by equations (58) and simplifying, we get

$$a = \frac{2L_1R_2 + 2L_2R_1 \pm \sqrt{4L_1^2R_2^2 - 4R_1R_2\left(\frac{L_1^2L_2}{L_1+L_2} - L_1^2 - 2L_1L_2\right)}}{8L_1L_2} \quad (74)$$

If we should neglect R_1 entirely, which, as a matter of fact, will introduce only a small error, since in practice R_1 is usually only about 1 per cent of R_2, then equation (74) will reduce to

$$a_2 = \frac{R_2}{2L_2} \quad \text{and} \quad a_1 = 0 \quad (75)$$

It is seen therefore that in the direct coupled circuit there is practically only one damping factor which is approximately the natural damping of the antenna.   It remains yet to evaluate the frequency constants in terms of the electrical constants of the circuits.   Referring back to equation (70) we have

$$N_1T_1 + N_2P_1 = 2P_1T_1a^2 - (P_1T_2 + P_2T_1)a + (T_1P_3 + P_1T_3)$$

$$N_1T_1 - N_2P_1 = (P_1T_2 - T_1P_2)a + (T_1P_3 - P_1T_3)$$

Introducing the values of P and T from equations (66) and neglecting the terms containing R_1 as a factor as being very

small compared with the other terms, we shall have

$$N_1 T_1 + N_2 P_1 = 4d_1(d_1 - a_1)\, a^2 + 4d_1 b_2 a - 8d_1$$

$$N_1 T_1 - N_2 P_1 = 4d_1 b_2 k_1 a + 8d_1 k_2$$

$$P_1 T_1 P_4 T_4 = -\frac{4d_1^2}{a_1}\, b_2^2\, (d_1 - a_1)$$

Introducing the above values in equation (70) we get

$$\lambda^2 = 4d_1(d_1 - a_1)\, a^2 + 4d_1 b_2 a - 8d_1 \pm \left. \frac{\sqrt{(4d_1 b_2 k_1 a + 8d_1 k_2)^2 - \frac{16d_1^2}{a_1}(d_1 - a_1)\, b_2^2 a^2}}{8d_1(d_1 - a_1)} \right\} \tag{76}$$

Introducing the values of a, b, and d as given by equations (58) and simplifying we get

$$\lambda^2 = L_1 C_1 \frac{L_2}{L_1 + L_2}\, a^2 - R_2 C_2 a + 2 \pm \left. \frac{\sqrt{\frac{L_1}{L_1 + L_2}(R_2 C_2 a - 2)^2 + \frac{L_2}{L_1 + L_2} R_2^2 C_2^2 a^2}}{2L_1 C_1 \frac{L_2}{L_1 + L_2}} \right\} \tag{77}$$

If we neglect the resistance of the antenna, the above equation will reduce to

$$\lambda^2 = \frac{2 \pm 2\sqrt{\dfrac{L_1}{L_1 + L_2}}}{2L_1 C_1 \dfrac{L_2}{L_1 + L_2}} = \frac{1}{L_1 C_1}\left\{\left(1 + \frac{L_1}{L_2}\right)\left(1 \pm \sqrt{\frac{L_2}{L_1 + L_2}}\right)\right\} \tag{78}$$

The value of λ^2 as given by equation (78) agrees with the results obtained by Seibt [1] who developed the same problem, but on the assumption that the resistance is negligible, which, of course, is a comparatively simple matter.

We have seen by equation (75) that in a direct coupled circuit practically only one damping factor exists, hence in this case the

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[1] Loc. cit.

equations corresponding to (17) for the electromagnetically coupled circuits will be

$$
\left.\begin{array}{l}
w_1 = A e^{-at} w_1' \\
w_2 = B e^{-at} w_2'
\end{array}\right\} \tag{79}
$$

and therefore equation (68) will reduce to

$$
\left.\begin{array}{l}
w_1 = e^{-at}\{K_1 e^{i\lambda_1 t} + K_2 e^{i\lambda_2 t}\} \\
w_2 = e^{-at}\{K_5 e^{i\lambda_1 t} + K_6 e^{i\lambda_2}\}
\end{array}\right\} \tag{80}
$$

The values of the potentials in the primary and antenna will be

$$
\left.\begin{array}{l}
V_1 = e^{-at}\{(m_1 K_1 + m_2 K_5) e^{i\lambda_1 t} + (m_1 K_2 + m_2 K_6) e^{i\lambda_2 t}\} \\
V_2 = e^{-at}\{(m_3 K_1 + m_4 K_5) e^{i\lambda_1 t} + (m_3 K_2 + m_4 K_6) e^{i\lambda_2 t}\}
\end{array}\right\} \tag{81}
$$

The expressions for the currents can be easily obtained from (81) by the aid of equation (55). The constants of integration can be evaluated by following the same method that was adopted for the case of electromagnetically coupled circuits.

CONCLUSION.

To summarize briefly the results obtained, we may say that for the case of electromagnetically coupled circuits, the equations (47)–(50) are the most complete expressions for the potential and current in the primary and secondary circuits. It will be seen that the potential and current in the primary and secondary circuits consist of two distinct oscillations, and each oscillation has two different damping factors. The expressions for the damping factors and frequency constants are given by equations (30), (31), and (32). If we shall put for brevity

$$
\frac{1}{\sqrt{L_1 C_1}} = p_1, \frac{1}{\sqrt{L_2 C_2}} = p_2
$$

then we find by referring to equation (32),

$$
\lambda_1^2 - \lambda_2^2 = \frac{\mu p_1 p_2}{1 - \mu^2} \tag{82}
$$

and from equation (30) we get

$$a_1 - a_2 = \frac{\sqrt{(\gamma_2 - \gamma_1 + 4\gamma_1\gamma_2\mu^2}}{2(1 - \mu^2)} \tag{83}$$

If μ is of such a magnitude that we may neglect μ^2 compared with unity, then we have

$$\left. \begin{array}{l} \lambda_1^2 - \lambda_2^2 = \mu p_1 p_2 \\[2em] a_1 - a_2 = \frac{1}{2}\sqrt{(\gamma_1 - \gamma_2)^2 + 4\gamma_1\gamma_2\mu^2} \end{array} \right\} \tag{84}$$

which shows that the difference between the two frequency constants varies more rapidly with the degree of coupling than that of the damping factors.

The constants of integration have been completely determined, and though the expressions for the constants are somewhat complicated, and it is a little difficult to see in which way they depend on the frequency constants and damping factors, yet their numerical evaluation in practical problems is comparatively simple.

In the case of direct coupled circuits, we have shown that the results obtained are in a general way analogous to those for the electromagnetically coupled circuits, except that one of the damping factors is very small and may be neglected. So in this case we have in each circuit two oscillations of different frequencies, and both have the same damping factor which is approximately the natural damping factor of the antenna itself.

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